

## Week 1 Worksheet Monday&Wednesday

Please inform your TA if you find any errors in the solutions.

### Topic: Integral Computing (including u-sub, trig identities and IBP)

Enhance the impression! What we've already learned:

$$\begin{aligned}\cos^2 \theta &= \frac{1+\cos(2\theta)}{2} & \sin^2 \theta &= \frac{1-\cos(2\theta)}{2} \\ \sin(2\theta) &= 2 \sin \theta \cos \theta & \int FG' dx &= FG - \int F' G dx\end{aligned}$$

$$1. \int 4 \cos(10x + 1) dx = \frac{2}{5} \sin(10x + 1) + C$$

$$\int e^{3x} dx = \frac{1}{3} e^{3x} + C$$

$$\int \sin(\pi x + 2017) dx = -\frac{1}{\pi} \cos(\pi x + 2017) + C$$

$$2. \int \cos^2(2x) \sin^2(2x) dx = \frac{1}{8}x - \frac{1}{64} \sin 8x + C$$

**Work:**

$$\begin{aligned}\int \cos^2(2x) \sin^2(2x) dx &= \int \left(\frac{\sin(4x)}{2}\right)^2 dx \\ &= \frac{1}{4} \int \sin^2(4x) dx \\ &= \frac{1}{8} \int 1 - \cos(8x) dx \\ &= \frac{1}{8} \left(x - \frac{\sin(8x)}{8}\right) + C\end{aligned}$$

$$3. \int \sin^3 x dx = \frac{\cos^3 x}{3} - \cos x + C$$

**Work:**

$$\begin{aligned}\int \sin^3 x dx &= \int \sin^2 x \sin x dx & u = \cos x, \quad du = -\sin x dx \\ &= - \int \sin^2 x du & \sin^2 x = 1 - \cos^2 x = 1 - u^2 \\ &= \int u^2 - 1 du \\ &= \frac{u^3}{3} - u + C \\ &= \frac{\cos^3 x}{3} - \cos x + C\end{aligned}$$

$$4. \int \arctan x dx = x \arctan x - \frac{1}{2} \ln |1 + x^2| + C$$

**Work:**

$$\begin{aligned}\int \arctan x dx &= x \arctan x - \int \frac{1}{1+x^2} x dx & \left| \begin{array}{l} F = \arctan x, \quad G' = 1 \\ F' = \frac{1}{1+x^2}, \quad G = x \end{array} \right. \\ &= x \arctan x - \frac{1}{2} \ln |1 + x^2| + C\end{aligned}$$

5.  $\int x^2 \cos x dx = x^2 \sin x + 2x \cos x - 2 \sin x + C$

**Work:**

$$\begin{aligned} \int x^2 \cos x dx &= x^2 \sin x - 2 \int x \sin x dx \\ &= x^2 \sin x - 2 \left( -x \cos x + \int \cos x dx \right) \\ &= x^2 \sin x + 2x \cos x - 2 \sin x + C \end{aligned}$$

$F = x^2, G' = \cos x$	$ $	$F' = 2x, G = \sin x$
$F = x, G' = \sin x$		$F' = 1, G = -\cos x$

6.  $\int \cos x \ln(\sin x) dx = \sin x \ln(\sin x) - \sin x + C$

**Work:**

$$\begin{aligned} \int \cos x \ln(\sin x) dx &= \sin x \ln(\sin x) - \int \cos x dx \\ &= \sin x \ln(\sin x) - \sin x + C \end{aligned}$$

$F = \ln(\sin x), G' = \cos x$	$ $	$F' = \frac{\cos x}{\sin x}, G = \sin x$
$F = \cos x, G' = \sin x$		$F' = 1, G = -\cos x$

7.  $\int e^z \sin(3z) dz = \frac{1}{10}e^z \sin 3z - \frac{3}{10}e^z \cos 3z + C$

**Work:**

$$\begin{aligned} \int e^z \sin(3z) dz &= e^z \sin(3z) - 3 \int \cos(3z) e^z dz \\ &= e^z \sin(3z) - 3 \int \cos(3z) e^z dz \\ &= e^z \sin(3z) - 3 \left( e^z \cos(3z) + 3 \int e^z \sin(3z) dz \right) \end{aligned}$$

$F = \sin(3z), G' = e^z$	$ $	$F' = 3 \cos(3z), G = e^z$
$F = \cos(3z), G' = e^z$		$F' = -3 \sin(3z), G = e^z$

Let  $I = \int e^z \sin(3z) dz$ , then

$$I = e^z \sin(3z) - 3e^z \cos(3z) - 9I$$

$$10I = e^z \sin(3z) - 3e^z \cos(3z)$$

$$I = \frac{1}{10} (e^z \sin(3z) - 3e^z \cos(3z)) + C$$

8. (a) Use the identity

$$\int x^n \cos x dx = x^n \sin x + nx^{n-1} \cos x - n(n-1) \int x^{n-2} \cos x dx$$

to compute  $\int x^2 \cos x dx$ . (Check that the answer is the same with the answer in Problem 5.)

**Solns:**

$$\begin{aligned} \int x^2 \cos x dx &= x^2 \sin x + 2x \cos x - 2 \int \cos x dx \\ &= x^2 \sin x + 2x \cos x - 2 \sin x + C \end{aligned}$$

(b) Show that

$$\int x^n \cos x dx = x^n \sin x + nx^{n-1} \cos x - n(n-1) \int x^{n-2} \cos x dx.$$

**Solns:**

$$\begin{aligned}\int x^n \cos x dx &= x^n \sin x - n \int x^{n-1} \sin x dx && \left| \begin{array}{ll} F = x^n & G' = \cos x \\ F' = nx^{n-1} & G = \sin x \end{array} \right. \\ &= x^n \sin x - n \left( -x^{n-1} \cos x + (n-1) \int x^{n-2} \cos x dx \right) && \left| \begin{array}{ll} F = x^{n-1} & G' = \sin x \\ F' = (n-1)x^{n-2} & G = -\cos x \end{array} \right. \\ &= x^n \sin x + nx^{n-1} \cos x - n(n-1) \int x^{n-2} \cos x dx\end{aligned}$$